HARVARD | BUSINESS | SCHOOL

9-185-066
REV: FEBRUARY 20, 2007

ANDRÉ F. PEROLD

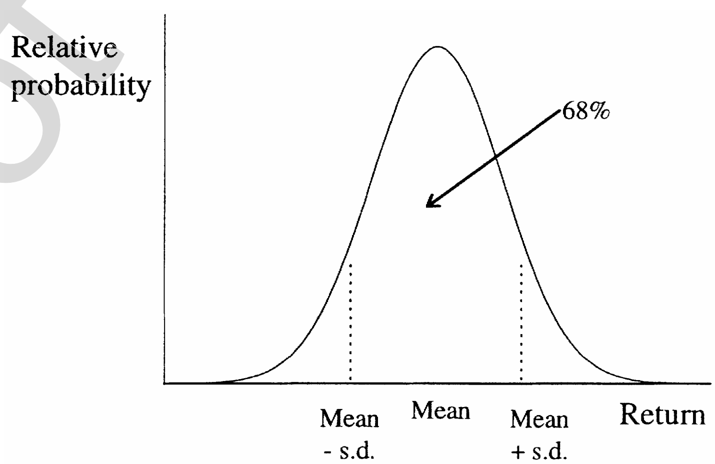
Introduction to Portfolio Theory

Portfolio theory is concerned with the risk-reducing role played by individual assets in an investment portfolio of several assets. The benefits of diversification were first formalized in 1952 by Harry Markowitz, who later was awarded the Nobel Prize in economics for this work. Portfolio theory is today a cornerstone of modern financial theory, as well as a widely used tool for managing risk-return tradeoffs in investment portfolios. This note examines the basic building blocks of the theory.

Means and Standard Deviations of Total Return

The return and risk of an asset are commonly measured in terms of the mean and standard deviation of total return, where total return represents income plus capital gains or losses. The mean is the return one expects to obtain on average; standard deviation is a measure of dispersion, in this case total volatility of return. For bell-shaped distributions (**Figure 1**), the return one actually experiences will fall within one standard deviation to either side of the mean about 68% of the time, within two standard deviations 95% of the time, and within three standard deviations 99.7% of the time.

Figure 1



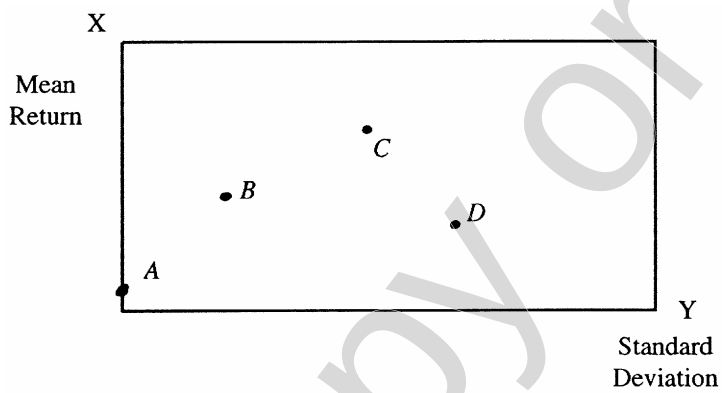
Professor André F. Perold prepared this note as the basis for class discussion.

[redacted]

The mean and standard deviation of return for a given asset can be computed from historical returns. In that case, however, they are merely summary descriptors of past performance, and may or may not reflect the probability distribution of future returns. Investors, of course, need to estimate the mean and standard deviation of future returns.

Assets can be compared on the basis of their means and standard deviations by drawing a simple diagram:

Figure 2



Asset *A* is a risk-free asset (i.e., cash or equivalent) since its standard deviation is zero. Asset *B* has a higher mean and higher standard deviation than *A*; asset *C* has a higher mean and standard deviation than *B*; and *D* has the highest standard deviation but a lower return than either *B* or *C*.

Investing in Only One Asset

If we could invest in only one of these four assets, which would we pick? The top left hand corner of **Figure 2** (marked X) is nirvana: much return, and no risk; the bottom right hand corner (marked Y) represents the worst of all worlds: no return, and much risk. As risk-averse investors, we prefer to own assets that are "closest" to X and "furthest away" from Y. For example, we can easily rule out asset *D* since both *B* and *C* have higher expected returns and lower standard deviations. Assets such as *D* are said to be *dominated* (by *B* and/or *C*). In Figure 2, *D* is the only dominated asset. Each of the others has either a higher expected return or a lower standard deviation than any other asset.

Dominated assets are relatively easy to remove from consideration. Choosing among undominated assets (here *A*, *B*, & *C*) is harder and requires knowledge of our risk tolerance. A risk-neutral investor would prefer *C*, and a very risk-averse investor would prefer *A*. Asset *B* is probably best for someone not overly risk-averse, and also not quite risk-neutral. In short, the choice among undominated assets is dictated by the degree to which the investor is personally willing to trade-off return for less risk.

Portfolios of a Riskless and a Risky Asset

What if we could put some of our money into cash (asset *A*) and the rest into one of *B*, *C*, or *D* (i.e., we can hold a portfolio of *A* and one other asset)? The reason for not investing in *D* is still valid: we can get higher return and less risk by investing in assets *B* or *C*. It turns out that the choice between *B* and *C* in the presence of a riskless asset can be based purely on dominance. This requires us to think about the means and standard deviations of all possible portfolios of cash and one other asset.

Suppose that

f = fraction invested in the risky asset (either *B* or *C*)

so that

$1 - f$ = fraction invested in cash.

Suppose further that

E_A, E_B, E_C and E_D = Expected returns on assets *A*, *B*, *C* and *D*, and

S_A (= 0), S_B, S_C *and* S_D = Standard deviations of return on assets *A*, *B*, *C* and *D*.

The formula for the expected return (E_P) of a portfolio of cash and another asset (say, *B*) is

$$E_P = (1 - f)E_A + fE_B$$

i.e., the portfolio expected return is simply the weighted average of the expected returns of the assets in the portfolio. If $f = 1$ (all our money in *B*) then $E_P = E_B$; if $f = 0$ (all our money in cash) then $E_P = E_A$.

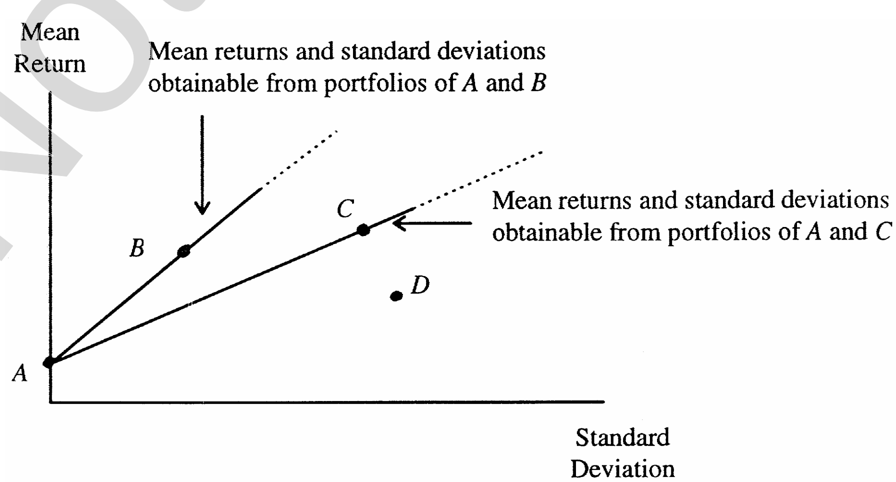
The formula for the standard deviation (S_P) of a portfolio of cash and asset *B* is:

$$S_P = fS_B$$

i.e., if we invest all our money in *B*($f = 1$) then $S_P = S_B$; if we invest half of our money in *B* and half in cash, then our risk is only half as large ($S_P = ½S_B$); and so on.

Both E_P and S_P are thus linearly related to the means and standard deviations of the assets in the portfolio. This is always true for E_P but only sometimes true for S_P, here because $S_A = 0$. Graphically, these relationships are as follows:

Figure 3



Each point on the line in **Figure 3** (say, the line through *A* and *B*) is the mean-standard deviation pair corresponding to a particular combination of *A* and *B*. For example, the point halfway between *A* and *B* gives the mean and standard deviation of a portfolio that is 50% invested in *A* and 50% invested in *B*. The lines extend beyond *B* and *C* by taking *f* to be larger than 1. This is possible if we can borrow at the rate of return on *A* (E_A) and leverage our investment in *B* or *C*.

Problems

1. Would you rather hold portfolios of *A* and *B* or *A* and *C*?

2. Whichever line (*A-B* or *A-C*) you choose to be on, how will you decide where to be on the line, i.e., what fraction of your money would you invest in the risky asset of your choice?

Portfolios of Two Risky Assets

Suppose we can invest our money in a portfolio of just two assets, *B* and *C*, both of which are risky. Let *f* be the fraction invested in *B* and 1 - *f* the fraction invested in *C*. As before, the portfolio return is just the weighted average:

$$E_P = fE_B + (1 - f)E_C$$

However, the formula for the portfolio standard deviation is now more complicated. It is given by

$$S_P = \sqrt{f^2 S_B^2 + 2f(1-f)RS_B S_C + (1-f)^2 S_C^2}$$

where *R* is the correlation between the returns on *B* and the returns on *C*.

Correlations are always between -1 and 1, and are a measure of the degree to which the returns on two assets fluctuate together. When $R = 1$, the returns are perfectly correlated (when one goes up, you know exactly by how much the other will go up); when $R = -1$, the returns are perfectly negatively correlated (when one goes up you know exactly by how much the other will go down). When $R = 0$, knowledge of what happened to one asset gives no information about the other. *R* is the same as the "*R*" in "*R squared*" from regression analysis.

The above square-root formula is the most general one, and covers all the special cases we have so far discussed. For example, when $f = 1$ (100% in *B*), the formula reduces to $S_P = S_B$; when $f = 0$ (100% in *C*), it reduces to $S_P = S_C$; when $S_C = 0$ (one asset is riskless), we obtain $S_P = fS_B$, the formula we saw earlier. Note further that when $R = 1$ (the assets are perfectly correlated), the formula can be manipulated algebraically and be rewritten as

$$S_P = fS_B + (1 - f)S_C$$

Thus, when the returns on assets *B* and *C* are perfectly positively correlated, the portfolio standard deviation is the weighted average of the standard deviations of the assets in the portfolio.

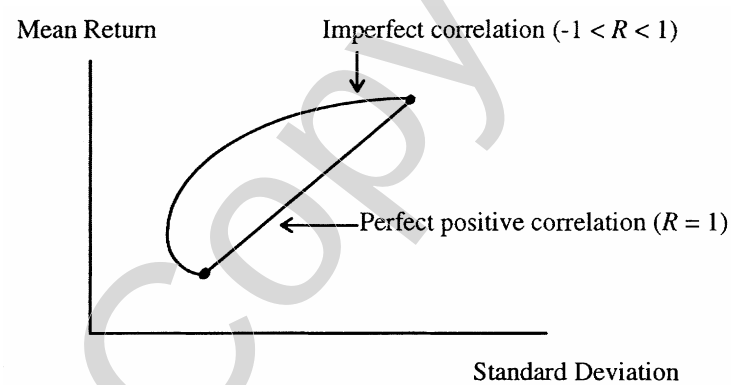
The important case, however, is when the assets are not perfectly correlated ($R < 1$). In this case, there is a curved relationship (nonlinear) between S_P and the fraction *f*. Moreover, S_P is always less than the weighted average of S_B and S_C.[1] This fact is of great significance and lies at the heart of portfolio theory. It means the following:

> If one holds, say, equal amounts of risky assets that are not perfectly correlated, the portfolio expected return will be the average of the expected returns of the assets in the portfolio. However, the portfolio standard deviation will be *less than* the average of the standard deviations of the assets in the portfolio.

The power of this statement is best demonstrated by imagining two assets with the same expected returns and the same standard deviations of return. By holding both assets in a portfolio, one obtains the same expected return as either one of them, but a standard deviation that is lower than any one of them individually. Thus, diversification leads to a reduction in risk without any sacrifice in expected return.

This relationship can be illustrated graphically as follows:

Figure 4

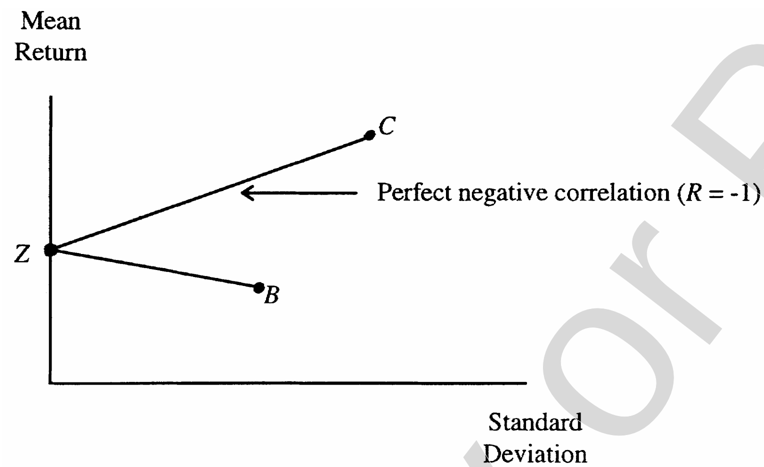


The extreme case occurs when *B* and *C* are perfectly negatively correlated (R = -1). Here, the means and standard deviations obtainable by combining *B* and *C* can be diagrammed as:

[1] This can be seen by algebraically manipulating the formula to get

$$S_P = \sqrt{(fS_B + (1-f)S_C)^2 - 2f(1-f)(1-R)S_B S_C}$$
$$S_P = \sqrt{(\text{average SD})^2 - \text{positive number}}$$
$$\text{i.e., } S_P < \text{average SD}$$

Figure 5



Here it is possible to have a portfolio of risky assets (*B* and *C*) that has zero risk (point *Z*) by judiciously choosing the fraction *f* of money invested in *B*.[2]

Problems (Continued)

3. Suppose you can invest only in risky assets *B* and *C* that are imperfectly correlated. Suppose further that the means and standard deviations obtainable by varying the fraction *f* of money invested in *B* (and hence the fraction 1 - *f* of money invested in *C*) are given by the curve in **Figure 4**. Which part of the curve is dominated and can hence be ruled out as undesirable? How would you choose where to be on the remaining part of the curve?

4. Suppose, in addition, that you can invest in a risk-free asset *A*. How does this affect the relative amounts you invest in *B* and *C*?

5. Can we still automatically rule out asset *D* (Figure 2) by dominance if we were allowed to hold it in a portfolio along with *A*, *B*, and *C*?

6. How would you generalize these ideas to three or more risky assets?

[2] This fraction can be shown to be $S_C/(S_B + S_C)$.